Today/this week: - recap
- ingredients to state the main thm.

Recollection:

Lecture 2: "Classical" deformation theory

Def'n: A pre-deformation functor is a functor

$$D : Art_{\mathbb{K}} \longrightarrow Set \quad \text{s.t.}$$

(1) $D(\mathbb{K}) \cong *$

def. functor:

(2) $\forall$
$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\sigma"] \\ C \arrow[r] & A \end{tikzcd}$$
the induced map

$$D(B \times_A C) \longrightarrow D(B) \times_{D(A)} D(C) \text{ is}$$

(a) surjective if σ is surjective
(b) bijective if $A = \mathbb{K}$

Here: • Artinian: local $\mathbb{K}$-algebra, f. dim'l as $\mathbb{K}$-v. sp.

• Condition (a) is about $\sigma : B \to A$ surjective

Recall: we decomposed such a surj. morphism into a finite composition of small morphisms.

$\sigma : B \to A$ is small if $(\ker \sigma) \cdot m_B = 0$
elementary if $(\ker \sigma)$ principal.

Example: $0 \longrightarrow (\varepsilon) \cong \mathbb{K} \xrightarrow{\cdot \varepsilon} \mathbb{K}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathbb{K}[\varepsilon]/(\varepsilon) \cong \mathbb{K} \to 0$

$0 \longrightarrow \mathbb{K} \xrightarrow{\cdot \varepsilon^n} \mathbb{K}[\varepsilon]/(\varepsilon^{n+1}) \longrightarrow \mathbb{K} \to 0$

Example: X locally Noetherian scheme $/\mathbb{K}$, p pt. in X

$\rightsquigarrow$ $R = \widehat{\mathcal{O}}_{X,p}$ complete local $\mathbb{K}$-algebra

$$h_R : \begin{cases} \mathrm{Art}_{\mathbb{K}} \longrightarrow \mathrm{Set} \\ A \longmapsto \mathrm{Hom}_{\mathbb{K}}(R,A) \end{cases}$$

h_R "pro-representable"

Recall: $T_D = D(\mathbb{K}[\varepsilon]/(\varepsilon^2))$ tangent vector space

$\mathcal{O}_D$ "obstruction space"

s.t. $\forall$ small extension $0 \to M \to B \xrightarrow{\alpha} A \to 0$

$\exists$ exact sequence (of ptd sets)

$$T_D \otimes M \to D(B) \xrightarrow{D(\alpha)} D(A) \xrightarrow{ob} \mathcal{O}_D \otimes M$$

$A = \mathbb{K}$ $\quad 0 \longrightarrow \quad$ —"—

Lecture 3 dgla's: get many examples!

$$\begin{cases} \mathrm{Lie}^{dg}_{\mathbb{K}} \longrightarrow \mathrm{FMP} \\ L \longmapsto \mathrm{Def}_L : \begin{cases} \mathrm{Art}_{\mathbb{K}} \longrightarrow \mathrm{Set} \\ A \longmapsto MC/_{gauge}(L \otimes A) \end{cases} \end{cases}$$

Ex: last week $\rightsquigarrow$ Kontsevich's formality

but:
- loses information: uses only L^0, L^1, L^2
- $\exists$ $L \underset{\text{quasi-iso}}{\simeq} L'$ $\rightsquigarrow$ $\mathrm{Def}_L \cong \mathrm{Def}_{L'}$ so is not an equivalence!

$\Longrightarrow$ evidence for higher cats

- given def problem, not clear how to "guess" the dgla
- infinitesimal def's correspond to

$$\begin{array}{ccc} \mathbb{K}[\varepsilon]/(\varepsilon^{n+1}) & \to & \mathbb{K}[\varepsilon]/(\varepsilon^n) \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{K} & \to & \mathbb{K} \oplus \mathbb{K}[1] \end{array}$$

model categories as source of $(\infty,1)$-categories

$\hookrightarrow$ transferred model structure

char $\mathbb{K} = 0$ field

- $\mathrm{Sym} : \mathrm{Vect}^{dg}_{\mathbb{K}} \rightleftarrows \mathrm{CAlg}^{dg}_{\mathbb{K}} : \mathcal{U}$ forget
- $f : \mathrm{Vect}^{dg}_{\mathbb{K}} \rightleftarrows \mathrm{Lie}^{dg}_{\mathbb{K}} : \theta$ forget

$\rightsquigarrow$ $(\infty,1)$-cats $\mathrm{CAlg}^{dg}_{\mathbb{K}}$, $\mathrm{Lie}^{dg}_{\mathbb{K}}$

equiv: quasi-iso
fibrations = level-wise surjections
cofibr. = degree-wise injections

Want analogs of Artinian, small, elementary, FMP in this context!

<u>Defn</u> Let $\mathrm{CAlg}^{aug}_{\mathbb{K}} := \mathrm{CAlg}^{dg}_{\mathbb{K}}/_{\mathbb{K}}$ be the category of <u>augmented</u> comm. dg algebras $/_{\mathbb{K}}$.

~~An element $A \in \mathrm{CAlg}^{dg}_{\mathbb{K}}/_{\mathbb{K}}$ is~~

① A morphism $\sigma : A' \to A$ in $\mathrm{CAlg}^{aug}_{\mathbb{K}}$ is <u>elementary</u> ~~small~~ if if there exists an integer + a homotopy pb

$$\begin{array}{ccc} A' & \longrightarrow & \mathbb{K} \\ \sigma \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & \mathbb{K} \oplus \mathbb{K}[n] \end{array}$$

(<u>Ex</u> inf. def.)

② A morphism $\sigma : A' \to A$ in $\mathrm{CAlg}^{aug}_{\mathbb{K}}$ is <u>small</u> if it is a finite composition of elementary ones.

③ An object $A \in \mathrm{CAlg}^{aug}_{\mathbb{K}}$ is <u>small</u> if $A \to \mathbb{K}$ is small.
"dg Artinian"

$\mathrm{CAlg}^{sm}_{\mathbb{K}} \subseteq \mathrm{CAlg}^{aug}_{\mathbb{K}}$ full sub-$(\infty,1)$-category

... does not have a model structure b/c does not have (co-)limits

These are analogs of small/Artinian b/c:

Prop 1. (4.17 [Porta]; 1.1.11 [Lu]) An object A in $\mathrm{CAlg}^{aug}_{\mathbb{K}}$ is small iff

(1) the homotopy groups $\pi_n(A)$ vanish if $n<0$ (=: "connective") and $n>>0$

(2) $\pi_n A$ is finite dim'l $/\mathbb{K}$ $\forall n$

(3) $\pi_0 A$ is local w/ max'l ideal m and $\mathbb{K} \to \pi_0 A/m$ is an isom.

($\Rightarrow \pi_0 A$ Artinian)

Prop 2. A morphism $\underbrace{\quad}_{\sigma: A' \to A}$ in $\mathrm{CAlg}^{aug}_{\mathbb{K}}$ is elementary iff

[P] Lemma 4.1.5; [Lu] Ex. 1.1.6

(1) $\mathrm{fib}(\phi) \simeq \mathbb{K}[n]$ for some $n \geq 0$ in A'-Mod

(2) if $n=0$, the map

$$\pi_0\, \mathrm{fib}(\phi) \otimes_{\pi_0 A'} \pi_0\, \mathrm{fib}(\phi) \longrightarrow \pi_0\, \mathrm{fib}(\phi)$$

vanishes

We defer the proofs to later or [Lurie], [Porta]

Prop 3 A morphism $\sigma: A' \to A$ in $\mathrm{CAlg}^{sm}_{\mathbb{K}}$ is small iff it induces a surjection of comm. rings $\pi_0(A') \to \pi_0(A)$

Idea of pf of Prop 1:

Postnikov tower of a cdga $A_{\leq 0}$, i.e. $\pi_i(A) \simeq 0$ for $i<0$
tower of fibrations in CAlg$_k$

$$A \to \cdots \to A_{\leq n+1} \to A_{\leq n} \to \cdots \to A_{\leq 1} \to A_{\leq 0} = \pi_0(A)$$

- $\pi_i(A_{\leq n}) \simeq 0$ for $i > n$ $\forall n$
- $\pi_i(A_{\leq n+1}) \simeq \pi_i(A_{\leq n})$ for $i \leq n$

$\Rightarrow$ exists up to isom., $A \simeq \lim_n A_{\leq n}$

If A satisfies conditions in proposition,
$A \simeq A_{\leq m}$ for $m >> 0$.
kernel K_{n+1} of $A_{\leq n+1} \to A_{\leq n}$ has $\pi_i(K_{n+1}) \simeq 0$ if $i \neq n+1$
$\simeq \pi_{n+1}(A)$ if $i = n+1$
$\Rightarrow A_{\leq n+1}$ is square-zero extension of $A_{\leq n}$ by $\pi_{n+1}(A)[n+1]$

<u>Def^n</u> A <u>formal moduli problem</u> is

a functor of $(\infty,1)$-categories
$$D : CAlg_{\mathbb{K}}^{sm} \longrightarrow sSet$$

st (1) $D(\mathbb{K}) \simeq pt$ (is a contractible space)

(2) if
$$(*)\quad \begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B' \arrow[d, "\sigma"] \\ A \arrow[r] & B \end{tikzcd}$$
is a homotopy pb st. σ is small

then
$$(*)\quad \begin{tikzcd} D(A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D(B') \arrow[d] \\ D(A) \arrow[r] & D(B) \end{tikzcd}$$
is hom. pb

<u>Remark:</u> Unravelling the definition, the condition on
$$\begin{tikzcd} A' \arrow[r] \arrow[d] & B' \arrow[d, "\sigma"] \\ A \arrow[r] & B \end{tikzcd}$$
reads:

- it is homotopy cartesian i.e. the natural morphism of cplxes $A' \longrightarrow (co)cone(A \oplus B' \longrightarrow B)$ is a quasi-iso
- σ induces a surjective morphism $\pi_0(B') \to \pi_0(B)$.

We then require that

- D sends quasi-isos of cdgas to a weak equiv. of sSet
- $D(\mathbb{K}) \simeq *$
-
$$\begin{tikzcd} D(A') \arrow[r] \arrow[d] & D(B') \arrow[d] \\ D(A) \arrow[r] & D(B) \end{tikzcd}$$
is homotopy ~~cartesian~~ pb of sSet

[Lu] Prop 1.1.15 + 1.1.19

Prop 4 Let $D : \mathrm{CAlg}_{\mathbb{K}}^{sm} \to s\,\mathrm{Set}$ be a functor. TFAE:

(A) D satisfies (2) above.

(B) If $(*)$ is a pb square st σ is elementary, then $D(*)$ is pb square

[P] Prop 4.2.5

(C) If $(*)$ is a pb square st σ is $\mathbb{K} \to \mathbb{K} \oplus \mathbb{K}[n]$ for some $n \geq 0$ then $D(*)$ is a pb square

(D) If $(*)$ is a pb sq. st the induced $\pi_0 B' \to \pi_0 B$ and $\pi_0 A \to \pi_0 B$ are surjective, then $D(*)$ is a pb square.

Rem: In (D), is equiv. to requiring one of the maps to be surj.

Uses:
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \end{tikzcd}$$
in $(\infty,1)$-cat. $\mathcal{C}$

If right sq. is pb, then the left square is pb iff outer —"—

$$\mathrm{FMP} \subseteq \mathrm{FMP}^{pre} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{sm}_{\mathbb{K}}, s\mathrm{Set})$$

full subcategory

as $(\infty,1)$-cat w/ level-wise equiv.

i.e. localization of this wrt w.e.

Constructions / Examples:

1. (pro) representable ones:

$A \in \mathrm{CAlg}^{aug}_{\mathbb{K}}$. $\quad \mathrm{Spf}\, A : \mathrm{CAlg}^{sm}_{\mathbb{K}} \longrightarrow s\mathrm{Set}$

$$\begin{tikzcd}
(\mathrm{Spf}\, A)(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(A,B) \arrow[d] \\
\{*\} \arrow[r, "\text{augm.}"'] & \mathrm{Map}(A,\mathbb{K})
\end{tikzcd}$$

is formal moduli problem: by def'n it is the mapping space in $\mathrm{CAlg}^{dg}_{\mathbb{K}}/_{\mathbb{K}}$ ← as $(\infty,1)$-cat. and commutes w/ holims therein.

(strictly speaking, not representable) ~~but pro-representable~~

Recall: $X = \mathrm{Spec}\, A$, A comm. $\mathbb{K}$-alg. of finite type,

$p : \mathrm{Spec}\, \mathbb{K} \to \mathrm{Spec}\, A \leftrightarrow p : A \to \mathbb{K}$ pt in X.

A has a Postnikov tower $\Rightarrow$ is pro-object

$\Rightarrow$ have FMP $\mathrm{Spf}\, A$.

Thm: $\mathrm{Spf}\, A \simeq \mathrm{Spf}\, \hat{A}_{p_*}$ $\widehat{\mathcal{O}_{X,p}}$

2) Construction by limits

$$F : I \longrightarrow \mathrm{Fun}(dCAlg_{\mathbb{K}}^{sm}, sSet)$$
$$\text{s.t. } F(i) \in FMP \quad \forall i \in I$$

$\Longrightarrow \underset{i}{\mathrm{holim}}\, F(i) \in FMP$ b/c holims commute with each other

i.e. $FMP \hookrightarrow FMP^{pr}$

admits holims, inclusion commutes w/ holims.

Ex: $\Omega_* D := * \underset{D}{\times} * \in FMP$

$$F = \begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & D \end{tikzcd}$$

i.e. if D is deformation problem of deforming some structure, $\Omega_* D$ is —"— —"— automorphisms of this structure.

3) ⇝ Toën, Sinan Yalin:

deformation of algebraic structure governed by an operad

E $\mathbb{Z}$graded v.sp.

$\mathcal{O}$ dg operad/$\mathbb{K}$

$$\mathcal{O}(E) : CAlg_{\mathbb{K}}^{sm} \longrightarrow sSet$$

$$\mathcal{O}(E)(A) := \underline{\mathrm{Map}}_{Op_{\mathbb{K}}}(\mathcal{O}, \underline{End}^{\otimes}(E) \otimes_{\mathbb{K}} A)$$

$$\sim \mathrm{Hom}_{Op_{\mathbb{K}}}(\tilde{\mathcal{O}}, \underline{End}^{\otimes}(E) \otimes A \otimes C^*(\Delta^\bullet))$$

cofibr replacement

eg $\mathcal{O} = As$. $\quad (\mathcal{O}(E)(\mathbb{K}) \neq \mathbb{K})$

	Def theory	Derived DT
test objects	$\mathrm{Art}_{\mathbb{K}}$ local Artinian $\mathbb{K}$-alg $\Updownarrow$ finite sequences of small ext. $\Updownarrow$ fin. seq of elementary ext. $\sigma: B \to A$ surj. = fin seq. of elem/small ext.	CAlg^{sm} $\Updownarrow$ finite seq of small ext. $\Updownarrow$ $\frac{1}{1}$ $\sigma: B \to A$ small = finite seq of elem.
(pre-)def functor:	$D: \mathrm{Art}_{\mathbb{K}} \longrightarrow \mathrm{Set}$ + $D(\mathbb{K}) = pt$	$D: \mathrm{CAlg}^{sm} \longrightarrow s\mathrm{Set}$ + $D(\mathbb{K}) \simeq pt$
	$D(B \times_A C) \to D(B) \times_{D(A)} D(C)$ - surj. if σ surj - bij if $A = \mathbb{K}$	$\begin{matrix} D(B \times_A C) & \longrightarrow & D(B) \\ \downarrow & \lrcorner & \downarrow \sigma \\ D(C) & \longrightarrow & D(A) \end{matrix}$ σ small
tangent	$T_D = D(k[\epsilon]/(\epsilon^2))$ $\mathcal{O}_D$?	tomorrow
smooth/ representable.	- - -	- - -
Ex:	$\widehat{\mathcal{O}}_{X,x}$	