

Today/this week:

- recap
- ingredients to state the main theorem.

①

Recollection:

Lecture 2: "Classical deformation theory"

Defn: A pre-deformation functor is a functor

$$D: \text{Art}_K \longrightarrow \text{Set} \quad \text{st.}$$

$$(1) D(K) \cong *$$

def. functor:

$$(2) \quad \forall \quad \begin{array}{ccc} B \times C & \longrightarrow & B \\ \downarrow \wedge & \searrow & \downarrow \sigma \\ C & \longrightarrow & A \end{array} \quad \text{the induced map}$$

$$D(B \times_C C) \longrightarrow D(B) \times_{D(A)} D(C) \quad \text{is}$$

(a) surjective if σ is surjective

(b) bijective if $A=K$

Here: • Artinian: local K -algebra, f. dim'l as K -v. sp.

• Condition (a) is about $\sigma: B \rightarrow A$ surjective

Recall: we decomposed such a surj. morphism into a finite composition of small morphisms.

$\sigma: B \rightarrow A$ is small if $(\ker \sigma) \cdot m_B = 0$

elementary if $(\ker \sigma)$ principal.

Example $0 \rightarrow (\epsilon) \cong K \xrightarrow{\cdot \epsilon} K[\epsilon]/(\epsilon^2) \rightarrow K[\epsilon]/(\epsilon) \cong K \rightarrow 0$

$$0 \rightarrow K \xrightarrow{\cdot \epsilon^n} K[\epsilon]/(\epsilon^{n+1}) \rightarrow K \rightarrow 0$$

Example: X locally Noetherian scheme / K , p pt. in X
 $\rightsquigarrow R = \hat{\mathcal{O}}_{X,p}$ complete local K -algebra

$$h_R \text{ of } \text{Art}_K \rightarrow \text{Set}$$

$$\left\{ \begin{array}{l} A \mapsto \text{Hom}_K(R, A) \end{array} \right.$$

h_R "pro-representable"

Recall: $T_D = D(K[[\epsilon]]/\langle \epsilon^2 \rangle)$ tangent vector space
 \mathcal{O}_D "obstruction space"

st. V small extension $0 \rightarrow M \rightarrow B \xrightarrow{\sim} A \rightarrow 0$

\exists exact sequence (of pd sets)

$$\textcircled{\otimes} \underbrace{T_D \otimes M} \rightarrow D(B) \xrightarrow{D(\alpha)} D(A) \xrightarrow{\text{ob}} \underbrace{\mathcal{O}_D \otimes M}$$

$A=K \quad 0 \rightarrow \quad \dashrightarrow$

Lecture 3 dgla's : get many examples!

$$\left\{ \begin{array}{l} \text{Lie}_K^{ds} \longrightarrow \text{FMP} \\ L \longmapsto \text{Def}_L: \left\{ \begin{array}{l} \text{Art}_K \rightarrow \text{Set} \\ A \mapsto \text{MC}_{\text{gauge}}(L \otimes A) \end{array} \right. \end{array} \right.$$

Ex: last week \rightsquigarrow Kontsevich's formality

but: - loses information: uses only L^0, L^1, L^2
 - $\exists L \cong L' \xrightarrow{\text{quasi-iso}} \text{Def}_L \cong \text{Def}_{L'}$ so is not an equivalence!

\implies evidence for higher cats

~~non-representability~~

- given def problem, not clear how to "guess" the dgla

- infinitesimal def's correspond to

$$\begin{array}{ccc} K[[\epsilon]]/\langle \epsilon^m \rangle & \rightarrow & K[[\epsilon]]/\langle \epsilon^n \rangle \\ \downarrow & \dashrightarrow & \downarrow \\ K & \rightarrow & K \oplus K[[\epsilon]] \end{array}$$

model categories as source of $(\infty, 1)$ -categories

↳ transferred model structure

char $K=0$ field

$\mathcal{M}od$

$$\bullet \text{ Sym: } \text{Vect}_{K}^{dg} \rightleftarrows \text{CAlg}_{K}^{dg} : \mathcal{U} \quad \text{forget}$$

$$\bullet f: \text{Vect}_{K}^{dg} \rightleftarrows \text{Lie}_{K}^{dg} : \emptyset \quad \text{forget}$$

$\rightsquigarrow (\infty, 1)$ -cats $\text{CAlg}_{K}^{dg}, \text{Lie}_{K}^{dg}$

equiv: quasi-iso

fibrations = level-wise surjections

cofibr. = degree-wise injections

Want analogs of Artinian, small, elementary, FMP in this context!

Defn Let $\text{CAlg}_{K}^{aug} / \text{CAlg}_{K}^{dg} / K$ be the category of augmented comm. dg algebras $/K$.

~~Artinian~~ ~~Artinian~~ ~~Artinian~~ $\text{CAlg}_{K}^{aug} / K$ vs

① A morphism $\sigma: A' \rightarrow A$ in CAlg_{K}^{aug} is elementary if there exists an integer n + a homotopy pb

$$\begin{array}{ccc} A' & \longrightarrow & K \\ \sigma \downarrow & \dashrightarrow & \downarrow \\ A & \longrightarrow & K \oplus K[n] \end{array} \quad (\text{Ex inf-def.})$$

② A morphism $\sigma: A' \rightarrow A$ in CAlg_{K}^{aug} is small if it is a finite composition of elementary ones.

③ An object $A \in \text{CAlg}_{K}^{aug}$ is small if $A \rightarrow K$ is small. "dg Artinian"

$\text{CAlg}_{\mathbb{K}}^{\text{sm}} \subseteq \text{CAlg}_{\mathbb{K}}^{\text{avg}}$ full sub-($\infty, 1$)-category

... does not have a model structure
b/c does not have
(co-)limits

These are analogs of small/Artinian b/c:

1.1.11 [Lu]

Prop 1. (4.17) [Porto] An object A in $\text{CAlg}_{\mathbb{K}}^{\text{avg}}$ is small iff

- (1) the homotopy groups $\pi_n(A)$ vanish if $n < 0$ (= "connective") and $n \gg 0$
- (2) $\pi_n A$ is finite dim'l / \mathbb{K} $\forall n$
- (3) $\pi_0 A$ is local w/ max'l ideal \mathfrak{m} and $\mathbb{K} \rightarrow \pi_0 A / \mathfrak{m}$ is an isom.

($\Rightarrow \pi_0 A$ Artinian)

Prop 2. A morphism $\sigma: A' \rightarrow A$ in $\text{CAlg}_{\mathbb{K}}^{\text{avg}}$ is elementary iff

- [P] (1) $\text{fib}(\phi) \cong \mathbb{K}[n]$ for some $n \geq 0$ in A' -Mod
- Lemma 1.1.5 (2) if $n=0$, the map $\pi_0 \text{fib}(\phi) \otimes_{\pi_0 A'} \pi_0 \text{fib}(\phi) \rightarrow \pi_0 \text{fib}(\phi)$ vanishes
- [Lu] Ex. 1.1.6

We defer the progs to later or [Lune], [Porto]

Prop 3 A morphism $\sigma: A \rightarrow A$ in $\text{CAlg}_{\mathbb{K}}^{\text{sm}}$ is small iff it induces a surjection of comm. rings $\pi_0(A') \rightarrow \pi_0(A)$



Idea of pf of Prop 1:

Postnikov tower of a cdga $A_{\leq 0}$, i.e. $\pi_i(A) \simeq 0$ for $i < 0$
tower of fibrations in cdgAlg_k

$$A \rightarrow \dots \rightarrow A_{\leq n+1} \rightarrow A_{\leq n} \rightarrow \dots \rightarrow A_{\leq 1} \rightarrow A_{\leq 0} = \pi_0 A$$

$$\bullet \pi_i(A_{\leq n+1}) \simeq 0 \text{ for } i > n \quad \forall n$$

$$\bullet \pi_i(A_{\leq n+1}) \simeq \pi_i(A_{\leq n}) \text{ for } i \leq n$$

\Rightarrow exists up to isom., $A \simeq \varinjlim_n A_{\leq n}$

If A satisfies conditions in proposition,
 $A \simeq A_{\leq m}$ for $m \gg 0$.

kernel K_{n+1} of $A_{\leq n+1} \rightarrow A_{\leq n}$ has $\pi_i(K_{n+1}) \simeq 0$ if $i \neq n+1$
 $\simeq \pi_{n+1}(A)$ if $i = n+1$

$\Rightarrow A_{\leq n+1}$ is square-zero extension
of $A_{\leq n}$ by $\pi_{n+1}(A)[n+1]$

Def'n A formal moduli problem is

a functor of $(\infty, 1)$ -categories

$$D: \text{CAlg}_{\mathbb{K}}^{\text{sm}} \longrightarrow \text{sSet}$$

st (1) $D(\mathbb{K}) \simeq \text{pt}$ (is a contractible space)

(2) if $A' \rightarrow B'$ is a homotopy pb st. σ is small

$$(*) \begin{array}{ccc} A' & \rightarrow & B' \\ \downarrow \sigma & & \downarrow \sigma \\ A & \rightarrow & B \end{array}$$

then $D(A') \rightarrow D(B')$

$$(*) \begin{array}{ccc} & \downarrow \sigma & \\ D(A') & \rightarrow & D(B') \\ & \downarrow \sigma & \\ D(A) & \rightarrow & D(B) \end{array} \text{ is hom. pb}$$

Remark: Unravelling the definition, the condition on

$$\begin{array}{ccc} A' & \rightarrow & B' \\ \downarrow & & \downarrow \sigma \\ A & \rightarrow & B \end{array} \text{ reads:}$$

- it is homotopy cartesian i.e. the natural morphism of cplxes $A' \rightarrow (\text{co} \text{cone}(A \oplus B' \rightarrow B))$ is a quasi-iso
- σ induces a surjective morphism $\pi_0(B') \rightarrow \pi_0(B)$.

We then require that

- D sends quasi-isos of cdgas to a weak equiv. of sSet
- $D(\mathbb{K}) \simeq *$
- $D(A') \rightarrow D(B')$

$$\begin{array}{ccc} & \downarrow & \\ D(A') & \rightarrow & D(B') \\ & \downarrow & \\ D(A) & \rightarrow & D(B) \end{array} \text{ is homotopy } \overset{\text{pb}}{\text{cartesian}} \text{ of sSet}$$

Prop 4 Let $D: \text{Cat}_K^{\text{sm}} \rightarrow \text{Set}$ be a functor. TFAE:

[Lw]

Prop 1.1.15
+ 1.1.19

[P] Prop 4.2.5

(A) D satisfies (2) above.

(B) If (x) is a pb square st α is elementary, then $D(x)$ is pb square.

(C) If (x) is a pb square st α is $K \rightarrow K \oplus K[n]$ for some $n \geq 0$

then $D(x)$ is a pb square

(D) If (x) is a pb sq. st. $\pi_0 A \rightarrow \pi_0 B$ and

$\pi_0 A \rightarrow \pi_0 B$ are surjective,

then $D(x)$ is a pb square.

Rem: In (D), is equiv. to requiring one of the maps to be surj.

Uses:

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z & \text{in } (\infty, 1)\text{-cat. } \mathcal{S} \\ \downarrow & & \downarrow & & \downarrow & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \end{array}$$

If right sq. is pb, then the left square is pb iff
outer \longleftarrow

$$\text{FMP} \subseteq \text{FMP}^{\text{pre}} \subseteq \text{Fun}(\text{CAlg}_{\mathbb{K}}^{\text{fin}}, \mathfrak{s}\text{Set})$$

↑
full subcategory

↑
as (co,1)-cat w/ level-wise equiv.

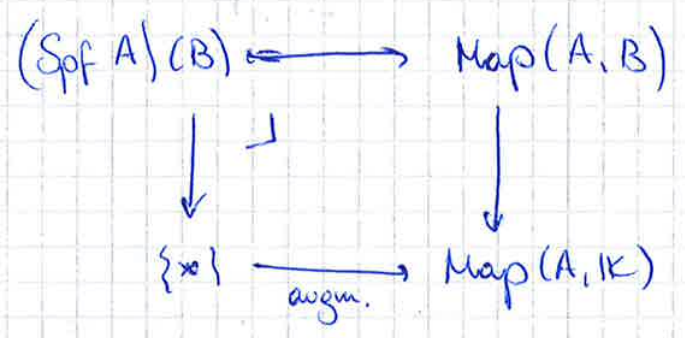
i.e. localization of this wrt w.e.

Constructions/Examples:

↳ (pro) representable ones:

$$A \in \text{CAlg}_{\mathbb{K}}^{\text{avg}}$$

$$\text{Spf } A: \text{CAlg}_{\mathbb{K}}^{\text{fin}} \rightarrow \mathfrak{s}\text{Set}$$



is formal moduli problem: by def'n it is the mapping space in $\text{CAlg}_{\mathbb{K}}^{\text{dg}} \leftarrow \text{as } (co,1)\text{-cat.}$ and commutes w/ holims therein.

(strictly speaking, not representable) ~~not/pro-representable~~

Recall: $X = \text{Spec } A$, A comm. \mathbb{K} -alg. of finite type, $p: \text{Spec } \mathbb{K} \rightarrow \text{Spec } A \leftarrow p^*A \rightarrow \mathbb{K}$ pt. in X .

A has a Postnikov tower \Rightarrow is pro-object \Rightarrow have FMP $\text{Spf } A$.

Thm: $\text{Spf } A \simeq \text{Spf } \widehat{A}_{P_{\leftarrow}} \widehat{\mathcal{O}_{X,P}}$

2) Construction by limits

$$F: I \longrightarrow \text{Fun}(\text{dCatg}_K^{\text{sm}}, \text{sSet})$$

s.t. $F(i) \in \text{FMP} \quad \forall i \in I$

$$\implies \text{holim}_i F(i) \in \text{FMP}$$

b/c holims commute with each other

i.e. $\text{FMP} \hookrightarrow \text{FMP}^{\text{pr}}$

admits holims, inclusion commutes w/ holims.

Ex: $\Omega_* D := * \times_{\mathbb{D}} * \in \text{PMF}$

$$F = \begin{array}{ccc} & * & \\ & \downarrow & \\ * & \longrightarrow & \mathbb{D} \end{array}$$

i.e. if D is deformation problem of deforming some structure, $\Omega_* D$ is $\text{---} \text{---} \text{---}$ automorphisms of this structure.

3) \rightsquigarrow Toën, Simon Yalin:

deformation of algebraic structure

governed by an operad

$E^{\mathbb{Z}}$ graded v.sp.

\mathcal{O} dg operad/ K

$$\mathcal{O}(E): \text{dCatg}_K^{\text{sm}} \longrightarrow \text{sSet}$$

$$\mathcal{O}(E)(A) := \text{Map}_{\text{Op}_K}(\mathcal{O}, \text{End}^{\otimes}(E) \otimes_K A)$$

$$\sim \text{Hom}_{\text{Op}_K}(\tilde{\mathcal{O}}, \text{End}^{\otimes}(E) \otimes A \otimes C^{\bullet}(\Delta^{\circ}))$$

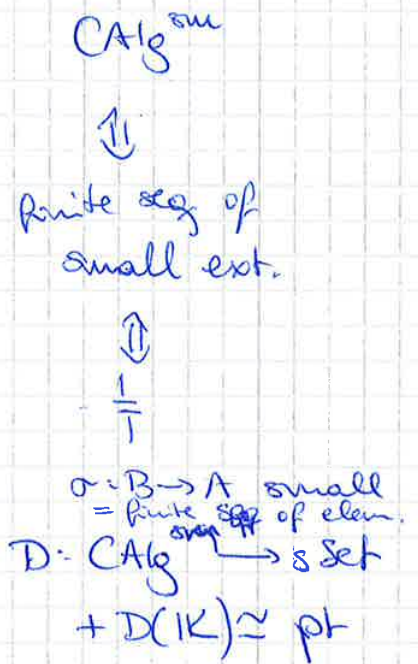
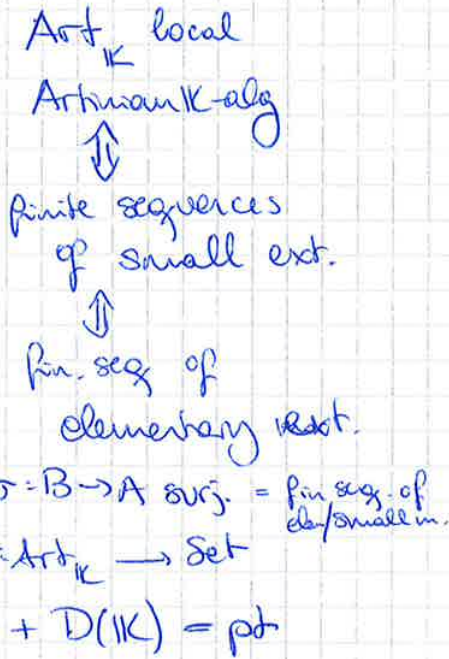
\uparrow
cofibr replacement

eg $\mathcal{O} = \text{As}$. $(\mathcal{O}(E)(K) \neq K)$

Def theory

Derived DT

test objects



(pre-)def functor:

$$D(B \times_A C) \rightarrow D(B) \times_{D(A)} D(C)$$

- surj. if σ surj
- bij. if $\sigma = A = K$

$$D(B \times_A C) \rightarrow D(B)$$

$$\downarrow \quad \downarrow$$

$$D(C) \rightarrow D(A)$$

σ small

tangent

$$T_D = D(\mathbb{R}[E])_{(E^2)}$$

\mathcal{O}_D ?

tomorrow

smooth/
representable.

Ex:

$$\hat{\mathcal{O}}_{x,x}$$